Summary on Units, Dimensions and Conversions on Electrodynamics

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Abstract. In this short note, we summarize five different units that have been traditionally used in electrodynamics. Among them, we focus on two most common unit systems, Gaussian and rationalized MKS. In addition, we discuss a special unit system in FLASH referred to as "none" in flash.par in which physical units are rescaled to further simplify MHD equations. Finally we seek for a conversion table among these three unit systems.

1. Basic Dimensions and Units

In discussing the units and dimensions, we choose the traditional choice of taking three independent dimensions as length (l), mass (m), and time (t).

1.1. Laws, relationships, and definitions

In table 1, we list several key physical quantities along with dimensions in MKS and Gaussian units. See more in (NRL 1994). Multiply the value in MKS units by the conversion factor to get the value of a quantity in Gaussian units. Multiples of 3 in the conversion factors result from the approximated speed of light $c = 2.9979 \times 10^{10}$ cm/sec $\approx 3 \times 10^{10}$ cm/sec.

Relationships & Laws

1. Continuity equation (charge conservation) for charge ρ: $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$.

2. Displacement $\mathbf{D}$: $\mathbf{D} = \epsilon \mathbf{E}$, where ϵ is permittivity.

3. Magnetic intensity $\mathbf{H}$: $\mathbf{B} = \mu \mathbf{H}$, where μ is permeability.

4. Coulomb's law on the force between two point charges q and q', separated by a distance r:
$$F_1 = k_1 \frac{qq'}{r^2}, \tag{1}$$
where the constant k_1 is a proportionality constant whose magnitude and dimensions either are determined by the equation if the magnitude and dimensions of the unit of charge have been specified independently or are chosen arbitrarily in order to define the unit of charge. Note that the unit of charge can be deduced from the relationship that the unit of force

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$(F = ma)$ is mlt^{-2}, and hence, by equating this with the right hand side of equation (1) the unit of charge is $m^{1/2}l^{3/2}t^{-1}$.

5. Ampére force law for steady-state magnetic phenomena is that the force per unit length between two infinitely long, parallel wires separated by a distance d and carrying currents I and I' is

$$\frac{dF_2}{dl} = 2k_2 \frac{II'}{d}. \tag{2}$$

6. It is easily checked that the ratio k_1/k_2 has the dimension of a velocity squared (l^2t^{-2}), and in fact, it can be found that $k_1/k_2 = c^2$, where c is the velocity of light.

7. From equations (9) and (10), the ratio of electric and magnetic fields E/B has the dimensions $l/t\alpha$.

8. Maxwell's equations:

$$\begin{aligned} -k_3 \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times \mathbf{E}, \ : \text{Faradays's Induction}, && (3) \\ 4\pi k_2 \alpha \mathbf{j} + \frac{k_2 \alpha}{k_1} \frac{\partial \mathbf{E}}{\partial t} &= \nabla \times \mathbf{B}, \ : \text{Ampére force law}, && (4) \\ \nabla \cdot \mathbf{E} &= 4\pi k_1 \rho, && (5) \\ \nabla \cdot \mathbf{B} &= 0. && (6) \end{aligned}$$

9. We find that

$$\frac{k_1}{k_2 k_3 \alpha} = c^2, k_3 = \frac{1}{\alpha}. \tag{7}$$

10. Generalized Ohm's law:

$$\mathbf{E} = -\frac{\mathbf{v} \times \mathbf{B}}{c} + \eta \mathbf{j} + \frac{1}{c} \frac{\mathbf{j} \times \mathbf{B}}{en_e} - \frac{\nabla p_e}{en_e}, \tag{8}$$

where $q = -e$ is electric charge, and n_e is the number of electrons per unit volume. For resistive Ohm's law, $\mathbf{E} = \eta \mathbf{j}$, or $\eta = \mathbf{E}/\mathbf{j}$.

11. In vacuum, $c = 1/\sqrt{\epsilon_0 \mu_0}$.

Definitions

1. Magnetic diffusivity (or magnetic viscosity) η is defined by electric conductivity σ: $\eta = 1/\mu_0 \sigma_0$.

2. Current I is defined by $I = dq/dt$, where q is electric charge. Thus the charge between times t_1 and t_2 is obtained by integrating both sides, $q = \int_{t1}^{t2} I dt$.

3. Permittivity ϵ is the measure of the resistance that is encountered when forming an electric field in a medium. It is a measure of how an electric field affects, and is affected by, a dielectric medium.

4. Permeability μ is the measure of the ability of a material to support the formation of a magnetic field within itself. It is a degree of magnetization that a material obtains in response to an applied magnetic field.

5. Electric field $\mathbf{E}$ of a point charge q is defined from the Coulomb's Law in equation (1) as the force per unit charge:

$$E = k_1 \frac{q}{r^2} \text{ (e.g., } E = \frac{qq'}{4\pi\epsilon r^2}\text{)}, \tag{9}$$

6. The magnetic field $\mathbf{B}$ is derived from the Ampére force law in equation (2) as being numerically proportional to the force per unit current with a proportionality constant α that may have certain dimensions chosen for convenience. Thus for a long straight wire carrying a current I, the magnetic induction $\mathbf{B}$ at a distance d has the magnitude (and dimensions)

$$B = 2k_2\alpha \frac{I}{d} \text{ (e.g., } B = \frac{\mu I}{2\pi r}\text{)}, \tag{10}$$

Physical Quantity	Symbol	Dimensions		MKS Units	Gaussian Units
		SI	Gaussian		
length	l	l	l	meter (m)	centimeter (cm)
time	t	t	t	second (sec)	second (sec)
mass	m	m	m	kilogram (kg)	gram (g)
(electric) charge	q	q	$m^{1/2}l^{3/2}t^{-1}$	coulomb	statcoulomb
displacement	$\mathbf{D}$	ql^{-2}	$m^{1/2}l^{-1/2}t^{-1}$	coulomb/m^2	statcoulomb/cm^2
charge density	ρ	ql^{-3}	$m^{1/2}l^{-3/2}t^{-1}$	coulomb/m^3	statcoulomb/cm^3
current	I	qt^{-1}	$m^{1/2}l^{3/2}t^{-2}$	ampere	statampere
current density	$\mathbf{j}$	$qt^{-1}l^{-2}$	$m^{1/2}l^{-1/2}t^{2}$	ampere/m^2	statampere/cm^2
electric field	$\mathbf{E}$	$mlt^{-2}q^{-1}$	$m^{1/2}l^{-1/2}t^{-1}$	volt/m	statvolt/cm
permittivity	ϵ	$t^2q^2m^{-1}l^{-3}$	1	farad/m	—
magnetic intensity	$\mathbf{H}$	$ql^{-1}t^{-1}$	$m^{1/2}l^{-1/2}t^{-1}$	ampere-turn/m	oersted
magnetic field	$\mathbf{B}$	$mt^{-1}q^{-1}$	$m^{1/2}l^{-1/2}t^{-1}$	tesla	gauss
permeability	μ	mlq^{-2}	1	henry/m	—
magnetic flux	Φ	$ml^2t^{-1}q^{-1}$	$m^{1/2}l^{3/2}t^{-1}$	weber	maxwell
electric conductivity	σ	$tq^2m^{-1}l^{-3}$	t^{-1}	siemens/m	sec^{-1}
resistivity	η	$t^{-1}q^{-2}ml^3$	t	ohm-m	sec
force	$\mathbf{F}$	mlt^{-2}	mlt^{-2}	newton	dyne
frequency	f	t^{-1}	t^{-1}	hertz	hertz
momentum	$\mathbf{p}$	mlt^{-1}	mlt^{-1}	kg-m/sec	kg-m/sec
thermal conductivity	κ	mlt^{-3}	mlt^{-3}	watt/m-deg (K)	erg/cm-sec-deg (K)
fluid viscosity	μ	$ml^{-1}t^{-1}$	$ml^{-1}t^{-1}$	kg/m-sec	poise

Table 1. Dimensions of physical quantities in MKS (SI) and Gaussian units.

x

System	k_1	k_2	k_3	α
Electrostatic (esu)	**1**	c^{-2}	**1**	1
Electromagnetic (emu)	c^2	**1**	**1**	1
Gaussian	**1**	c^{-2}	c^{-1}	c
Heavyside-Lorentz	$\frac{1}{4\pi}$	$\frac{1}{4\pi c^2}$	c^{-1}	c
Rationalized MKS	$\frac{1}{4\pi\epsilon_0}(=\frac{c^2}{10^7})$	$\frac{\mu_0}{4\pi}=\frac{1}{10^7}$	**1**	1

Table 2. Magnitudes and dimensions of the electromagnetic constants for five systems of units. Values in red are two constant values that can (and must) be chosen arbitrarily, and the others can be defined accordingly. Two systems in blue, Gaussian and rationalized MKS, are the most common and popular choices of systems.

System	ϵ_0	μ_0
Electrostatic (esu)	**1**	c^{-2}
Electromagnetic (emu)	c^{-2}	**1**
Gaussian	1	1
Heavyside-Lorentz	1	1
Rationalized MKS	$\frac{10^7}{4\pi c^2}$	$\frac{4\pi}{10^7}$

Table 3. Definitions of ϵ_0 and μ_0 in five systems of units. Values in red are constant values that can (and must) be chosen arbitrarily.

2. Conversions

In this section we present tables of conversion in which factors are to be multiplied to convert one system to the other. We note that there is a special system referred to as “none” in FLASH. This system rescales the magnetic field (thus the derived quantity electric field implicitly) in such a way that factors of 4π and the speed of light c are absorbed into the physical variables. As a result, they simply can be substituted by unity. Of course, this doesn't mean that $\pi = 1/4$ at all! The advantage of this “none” unit system in FLASH is to provide a very simplified set of Maxwell's equations in their final form.

Three different choices for scaling **B** field are considered: none, cgs (same as Gaussian) and SI (same as MKS). Among three of them, the conversion between “none” to “Gaussian” is of our most interest in FLASH. And we consider each case now.

2.1. unitSystem=“none” in flash.par

It is the most convenient unit system that simplifies the Maxwell's equation without appearing μ_0, ϵ_0, and 4π, while providing all physical variables in cgs unit. In other words, in this unit, those electromagnetic variables are still in cgs (but not (cgs) Gaussian!) without μ_0, ϵ_0, and 4π. One thing to keep in mind is that, in order to specify an initial value for a magnetic diffusivity η for resistive MHD, users should give a value which is in the same unit as diffusivity constants (e.g., kinematic viscosity) that have units of l^2t^{-1}. This is because that the value η is used in Diffuse_computeDt.F90 which expects to have all diffusivity constants in the units of l^2t^{-1}.

Unit check: $\eta_{\rm code}(l^2t^{-1}) = \eta_{\rm flash.par}(l^2t^{-1})$. Expressions in paranthesis are units for each.

2.2. unitSystem="cgs" in flash.par

Quantity	FLASH's None	Gaussian
magnetic field	$\mathbf{B}$	$\frac{\mathbf{B}}{\sqrt{4\pi}}$
electric field	$\mathbf{E}$	$\frac{c\mathbf{E}}{\sqrt{4\pi}}$
current density	$\mathbf{j}$	$\frac{\sqrt{4\pi}}{c}\mathbf{j}$
vector potential	$\mathbf{A}$	$\frac{\mathbf{A}}{\sqrt{4\pi}}$
magnetic diffusivity (aka magnetic viscosity, or resistivity)	η	$\frac{c^2}{4\pi}\eta$

Table 4. Conversion table for electrodynamics quantities from FLASH's none to Gaussian.

In this case, FLASH explicitly converts magnetic field ($\mathbf{B}$) and magnetic diffusivity constant (η), expecting the initial values given by users are in cgs: $m^{1/2}l^{-1/2}t^{-1}$ for $\mathbf{B}$, and t for η.

Unit check: $\eta_{\rm code}(l^2t^{-1}) = \frac{c^2}{4\pi}(l^2/t^{-2}) \times \eta_{\rm flash.par}(t)$. Expressions in paranthesis are units for each.

2.3. unitSystem="SI" in flash.par

Quantity	Gaussian	Rationalized MKS
speed of light	c	$\frac{1}{\sqrt{\mu_0\epsilon_0}}$
magnetic field	$\mathbf{B}$	$\sqrt{\frac{4\pi}{\mu_0}}\mathbf{B}$
electric field	$\mathbf{E}$	$\sqrt{4\pi\epsilon_0}\mathbf{E}$
current density	$\mathbf{j}$	$\frac{\mathbf{j}}{\sqrt{4\pi\epsilon_0}}$
vector potential	$\mathbf{A}$	$\sqrt{\frac{4\pi}{\mu_0}}\mathbf{A}$
magnetic diffusivity (aka magnetic viscosity, or resistivity)	η	$4\pi\epsilon_0\eta$

Table 5. Conversion table for electrodynamics quantities from Gaussian to Rationalized MKS.

For this case, it is important to make sure that users also convert all the other physical (gas dynamics) variables to SI as well. FLASH additionally explic-

Quantity	FLASH's None	Rationalized MKS
magnetic field	$\mathbf{B}$	$\frac{\mathbf{B}}{\sqrt{\mu_0}}$
electric field	$\mathbf{E}$	$c\sqrt{\epsilon_0}\mathbf{E}$
current density	$\mathbf{j}$	$\frac{\mathbf{j}}{c\sqrt{\epsilon_0}}$
vector potential	$\mathbf{A}$	$\frac{\mathbf{A}}{\sqrt{\mu_0}}$
magnetic diffusivity (aka magnetic viscosity, or resistivity)	η	$\frac{10^7}{4\pi}\eta = \frac{1}{\mu_0}\eta$

Table 6. Conversion table for electrodynamics quantities from FLASH's none to Rationalized MKS.

itly converts magnetic field ($\mathbf{B}$) and magnetic diffusivity constant (η), expecting the initial values given by users are in SI: $mt^{-1}q^{-1}$ for $\mathbf{B}$, and $t^{-1}q^{-2}ml^3$ for η.

Unit check: $\eta_{\text{code}}(l^2t^{-1}) = \frac{10^7}{4\pi}(m^{-1}l^{-1}q^2) \times \eta_{\text{flash.par}}(t^{-1}q^{-2}ml^3)$. Expressions in paranthesis are units for each. Note that $\mu_0 = 4\pi \times 10^{-7}$ henry/meter.

3. Remarks: original argument by Mateusz Ruszkowski

Consider a case in FLASH that if unitSystem="none" is chosen in flash.par. Let B_x be initialized by B_{x0} (or $B_{\text{flash.par}}$) and B_y and B_z are all zeros in flash.par. What is the initial strength B of $\mathbf{B}$ field in gauss for this initial condition? The answer is $\sqrt{4\pi}B_{x0}$.

To prove, note that using units = "none" means that the magnetic field specified in flash.par is not rescaled in any way by the code. So, for example, if we want to setup an equipartition field, then we would take $B = \sqrt{8\pi\rho c_{\text{is}}^2}$ gauss, calculated by some theorist who knows nothing about the inner workings of the code, put it in flash.par, and this would be the field that the code would plug into its equations (c_{is} is a sound speed in isothermal gas). Now, since the magnetic pressure in the code is $p_B = B^2/2$, this means that, in the code, $p_B = 4\pi p_{\text{gas}}$, which is not an equipartition situation. So for units="none", the magnetic field in flash.par is not really in gauss. Instead, the actual field strength B in gauss is $\sqrt{4\pi}B_{x0}$. Same goes for the field that is in the FLASH output files, i.e., the actual field strength in gauss is $\sqrt{4\pi}B_{\text{chkpoint}}$.

4. Example

Let's have an example of solving the Dai and Woodward shock tube problem (JCP, 1994) in Fig 8 using FLASH. Note that their governing equations have factors of $\sqrt{4\pi}$ for magnetic fields, whereas FLASH doesn't. Therefore in order to initialize the problem that is in Gausian cgs unit using FLASH's none unit, one has to divide magnetic fields by $\sqrt{4\pi}$ to specify initial conditions.

When plotting, use the conversion of multiplying the fields by $\sqrt{4\pi}$, that is $\sqrt{4\pi}B_{\mathrm{chkpoint}}$, in order to compare the FLASH's results with the results in the paper.

References

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